Local scale invariance, conformal invariance and dynamical scaling

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Abstract. Building on an analogy with conformal invariance, local scale transformations consistent with dynamical scaling are constructed. Two types of local scale invariance are found which act as dynamical space-time symmetries of certain non-local free field theories. Physical applications include uniaxial Lifshitz points and ageing in simple ferromagnets.

Scale invariance is a central notion of modern theories of critical and collective phenomena. We are interested in systems with strongly anisotropic or dynamical criticality. In these systems, two-point functions satisfy the scaling form

$$G(t, \boldsymbol{r}) = b^{2x} G(b^\theta t, b\boldsymbol{r}) = t^{-2x/\theta} \Phi\left(rt^{-1/\theta}\right) = r^{-2x} \Omega\left(tr^{-\theta}\right) \tag{1}$$

where t stands for 'temporal' and $\boldsymbol{r}$ for 'spatial' coordinates, x is a scaling dimension, θ the anisotropy exponent (when t corresponds to physical time, $\theta = z$ is called the dynamical exponent) and Φ, Ω are scaling functions. Physical realizations of this are numerous, see [1] and references therein. For isotropic critical systems, $\theta = 1$ and the 'temporal' variable t becomes just another coordinate. It is well-known that in this case, scale invariance (1) with a constant rescaling factor b can be replaced by the larger group of conformal transformations $b = b(t, \boldsymbol{r})$ such that angles are preserved. It turns out that in the case of one space and one time dimensions, conformal invariance becomes an important dynamical symmetry from which many physically relevant conclusions can be drawn [2].

Given the remarkable success of conformal invariance descriptions of equilibrium phase transitions, one may wonder whether similar extensions of scale invariance also exist when $\theta \neq 1$. Indeed, for $\theta = 2$ the analogue of the conformal group is known to be the Schrödinger group [3, 4] (and apparently already known to Lie). While applications of the Schrödinger group as dynamical space-time symmetry are known [5], we are interested here in the more general case when $\theta \neq 1, 2$. We shall first describe the construction of these *local scale transformations*, show that they act as a dynamical symmetry, then derive the functions Φ, Ω and finally comment upon some physical applications. For details we refer the reader to [6].

The defining axioms of our notion of *local scale invariance* from which our results will be derived, are as follows (for simplicity, in $d = 1$ space dimensions).

(i) We seek space-time transformations with infinitesimal generators X_n, such that time undergoes a Möbius transformation

$$t \rightarrow t' = \frac{\alpha t + \beta}{\gamma t + \delta} \ ; \ \alpha\delta - \beta\gamma = 1 \tag{2}$$

and we require that even after the action on the space coordinates is included, the commutation relations

$$[X_n, X_m] = (n - m)X_{n+m} \tag{3}$$

remain valid. This is motivated from the fact that this condition is satisfied for both conformal and Schrödinger invariance.

(ii) The generator X_0 of scale transformations is

$$X_0 = -t\partial_t - \frac{1}{\theta}r\partial_r - \frac{x}{\theta} \tag{4}$$

with a scaling dimension x. Similarly, the generator of time translations is $X_{-1} = -\partial_t$.

(iii) Spatial translation invariance is required.

(iv) Since the Schrödinger group acts on wave functions through a projective representation, generalizations thereof should be expected to occur in the general case. Such extra terms will be called *mass terms*. Similarly, extra terms coming from the scaling dimensions should be present.

(v) The generators when applied to a two-point function should yield a finite number of independent conditions, i.e. of the form $X_n G = 0$.

Proposition 1: *Consider the generators*

$$X_n = -t^{n+1}\partial_t - \sum_{k=0}^{n} \binom{n+1}{k+1} A_{k0} r^{\theta k+1} t^{n-k} \partial_r - \sum_{k=0}^{n} \binom{n+1}{k+1} B_{k0} r^{\theta k} t^{n-k} \tag{5}$$

where the coefficients A_{k0} and B_{k0} are given by the recurrences $A_{n+1,0} = \theta A_{n0} A_{10}$, $B_{n+1,0} = \frac{\theta}{n-1}(nB_{n0}A_{10} - A_{n0}B_{10})$ for $n \geq 2$ where $A_{00} = 1/\theta$, $B_{00} = x/\theta$ and in addition one of the following conditions holds: (a) $A_{20} = \theta A_{10}^2$ (b) $A_{10} = A_{20} = 0$ (c) $A_{20} = B_{20} = 0$ (d) $A_{10} = B_{10} = 0$. These are the most general linear first-order operators in ∂_t and ∂_r consistent with the above axioms (i) and (ii) and which satisfy the commutation relations $[X_n, X_{n'}] = (n - n')X_{n+n'}$ for all $n, n' \in \mathbb{Z}$.
Closed but lengthy expressions of the X_n for all $n \in \mathbb{Z}$ are known [6]. In order to include space translations, we set $\theta = 2/N$ and use the short-hand $X_n = -t^{n+1}\partial_t - a_n\partial_r - b_n$. We then define

$$Y_m = Y_{k-N/2} = -\frac{2}{N(k+1)}\left(\frac{\partial a_k(t,r)}{\partial r}\partial_r + \frac{\partial b_k(t,r)}{\partial r}\right) \tag{6}$$

where $m = -\frac{N}{2} + k$ and k is an integer. Clearly, $Y_{-N/2} = -\partial_r$ generates space translations.
Proposition 2: *The generators X_n and Y_m defined in eqs. (5,6) satisfy the commutation relations*

$$[X_n, X_{n'}] = (n - n')X_{n+n'} \quad , \quad [X_n, Y_m] = \left(n\frac{N}{2} - m\right)Y_{n+m} \tag{7}$$

in one of the following three cases: (i) B_{10} arbitrary, $A_{10} = A_{20} = B_{20} = 0$ and N arbitrary. (ii) B_{10} and B_{20} arbitrary, $A_{10} = A_{20} = 0$ and $N = 1$. (iii) A_{10} and B_{10} arbitrary, $A_{20} = A_{10}^2$, $B_{20} = \frac{3}{2}A_{10}B_{10}$ and $N = 2$.
In each case, the generators depend on two free parameters. The physical interpretation of the free constants $A_{10}, A_{20}, B_{10}, B_{20}$ is still open. In the cases (ii) and (iii), the generators close into a Lie algebra, see [6] for details. For case (i), a closed Lie algebra exists if $B_{10} = 0$.

Turning to the mass terms, we now restrict to the projective transformations in time, because we shall only need those in the applications later. It is enough to give merely the 'special' generator X_1 which reads for $B_{10} = 0$ as follows [6]

$$X_1 = -t^2\partial_t - Ntr\partial_r - Nxt - \alpha r^2\partial_t^{N-1} - \beta r^2\partial_r^{2(N-1)/N} - \gamma\partial_r^{2(N-1)/N}r^2 \tag{8}$$

where α, β, γ are free parameters (the cases (ii,iii) of Prop. 2 do not give anything new). Furthermore, it turns out that the relation $[X_1, Y_{N/2}] = 0$ for N integer is only satisfied in one of the two cases (I) $\beta = \gamma = 0$ which we call *Type I* and (II) $\alpha = 0$ which we call *Type II*. In both cases, all generators can be obtained by repeated commutators of $X_{-1} = -\partial_t$,

$Y_{-N/2} = -\partial_r$ and X_1, using (7). Commutators between two generators Y_m are non-trivial and in general only close on certain 'physical' states. One might call such a structure a *weak Lie algebra*. These results depend on the construction [6] of *commuting* fractional derivatives satisfying the rules $\partial_r^{a+b} = \partial_r^a \partial_r^b$ and $[\partial_r^a, r] = a\partial_r^{a-1}$ (the standard Riemann-Liouville fractional derivative is not commutative, see e.g. [7]).

For $N = 1$, the generators of both Type I and Type II reduce to those of the Schrödinger group. For $N = 2$, Type I reproduces the well-known generators of $2D$ conformal invariance (without central charge) and Type II gives another infinite-dimensional group whose Lie algebra is isomorphic to the one of $2D$ conformal invariance [6].

Dynamical symmetries can now be discussed as follows, by calculating the commutator of the 'Schrödinger-operator' $\mathcal{S}$ with X_1. We take $d = 1$ and $B_{10} = 0$ for simplicity.

Proposition 3: *The realization of Type I sends any solution $\psi(t, r)$ with scaling dimension $x = 1/2 - (N-1)/N$ of the differential equation*

$$\mathcal{S}\psi(t, \boldsymbol{r}) = \left(-\alpha\partial_t^N + \left(\frac{N}{2}\right)^2 \partial_r^2\right)\psi(t, r) = 0 \qquad (9)$$

into another solution of the same equation.

Proposition 4: *The realization of Type II sends any solution $\psi(t, r)$ with scaling dimension $x = (\theta - 1)/2 + (2 - \theta)\gamma/(\beta + \gamma)$ of the differential equation*

$$\mathcal{S}\psi(t, r) = \left(-(\beta + \gamma)\partial_t + \frac{1}{\theta^2}\partial_r^\theta\right)\psi(t, r) = 0 \qquad (10)$$

into another solution of the same equation.

In both cases, $\mathcal{S}$ is a Casimir operator of the 'Galilei'-subalgebra generated from $X_{-1}, Y_{-N/2}$ and the generalized Galilei-transformation $Y_{-N/2+1}$. The equations (9,10) can be seen as equations of motion of certain free field theories, where x is the scaling dimension of that free field ψ. These free field theories are non-local, unless N or θ are integers, respectively.

From a physical point of view, these wave equations suggest that the applications of Types I and II are very different. Indeed, eq. (9) is typical for equilibrium systems with a scaling anisotropy introduced through competing uniaxial interactions. Paradigmatic cases of this are so-called Lifshitz points which occur for example in magnetic systems when an ordered ferromagnetic, a disordered paramagnetic and an incommensurate phase meet (see [8] for a recent review). On the other hand, eq. (10) is reminiscent of a Langevin equation which may describe the temporal evolution of a physical system. In any case, causality requirements can only be met by an evolution equation of first order in ∂_t.

Next, we find the scaling functions Φ, Ω in eq. (1) from the assumption that G transforms covariantly under local scale transformations.

Proposition 5: *Local scale invariance implies that for Type I, the function $\Omega(v)$ must satisfy*

$$\left(\alpha\partial_v^{N-1} - v^2\partial_v - Nx\right)\Omega(v) = 0 \qquad (11)$$

together with the boundary conditions $\Omega(0) = \Omega_0$ and $\Omega(v) \sim \Omega_\infty v^{-Nx}$ for $v \to \infty$. For Type II, we have

$$\left(\partial_u + \theta(\beta + \gamma)u\partial_u^{2-\theta} + 2\theta(2 - \theta)\gamma\partial_u^{1-\theta}\right)\Phi(u) = 0 \qquad (12)$$

with the boundary conditions $\Phi(0) = \Phi_0$ and $\Phi(u) \sim \Phi_\infty u^{-2x}$ for $u \to \infty$.

Here $\Omega_{0,\infty}$ and $\Phi_{0,\infty}$ are constants. The ratio β/γ turns out to be universal and related to x. From the linear differential equations (11,12) the scaling functions $\Omega(v)$ and $\Phi(u)$ can be found explicitly using standard methods [6].

Given these explicit results, the idea of local scale invariance can be tested in specific models. Indeed, the predictions for $\Omega(v)$ coming from Type I with $N = 4$ nicely agree with

cluster Monte Carlo data for the spin-spin and energy-energy correlators of the $3D$ ANNNI model at its Lifshitz point [6, 9]. On the other hand, the predictions of Type II have been tested extensively in the context of ageing ferromagnetic spin systems to which we turn now.

Consider a ferromagnetic spin system (e.g. an Ising model) prepared in a high-temperature initial state and then quenched to some temperature T at or below the critical temperature T_c. Then the system is left to evolve freely (for recent reviews, see [10, 11]). It turns out that clusters of a typical time-dependent size $L(t) \sim t^{1/z}$ form and grow, where z is the dynamical exponent. Furthermore, two-time observables such as the response function $R(t, s; \boldsymbol{r} - \boldsymbol{r}') = \delta\langle\sigma_{\boldsymbol{r}}(t)\rangle/\delta h_{\boldsymbol{r}'}(s)$ depend on *both* t and s, where $\sigma_{\boldsymbol{r}}$ is a spin variable and $h_{\boldsymbol{r}'}$ the conjugate magnetic field. This breaking of time-translation invariance is called *ageing*. We are mainly interested in the autoresponse function $R(t, s) = R(t, s; \boldsymbol{0})$. One finds a dynamic scaling behaviour $R(t, s) \sim s^{-1-a}f_R(t/s)$ with $f_R(x) \sim x^{-\lambda_R/z}$ for $x \gg 1$ and where λ_R and a are exponents to be determined.

In order to apply local scale invariance to this problem, we must take into account that time translation invariance does *not* hold. The simplest way to do this is to remark that the Type II-subalgebra spanned by X_0, X_1 and the Y_m leaves the initial line $t = 0$ invariant, see (8). Therefore the autoresponse function $R(t, s)$ is fixed by the two covariance conditions $X_0 R = X_1 R = 0$. Solving these differential equations and comparing with the above scaling forms leads to [6]

$$R(t, s) = r_0 \, (t/s)^{1+a-\lambda_R/z} \, (t - s)^{-1-a} \quad , \quad t > s \tag{13}$$

where r_0 is a normalization constant. Therefore the functional form of R is completely fixed once the exponents a and λ_R/z are known. Similarly, the spatio-temporal response $R(t, s; \boldsymbol{r}) = R(t, s)\Phi\left(r(t - s)^{-1/z}\right)$, with the scaling function $\Phi(u)$ determined by (12).

The prediction (13) has been confirmed recently in several physically distinct systems undergoing ageing, see [12, 13, 14, 15] and references therein. These confirmations (which go beyond free field theory) suggest that (13) should hold independently of (i) the value of the dynamical exponent z (ii) the spatial dimensionality $d > 1$ (iii) the numbers of components of the order parameter and the global symmetry group (iv) the spatial range of the interactions (v) the presence of spatially long-range initial correlations (vi) the value of the temperature T (vii) the presence of weak disorder. Evidently, additional model studies are called for to test this conjecture further.

Summarizing, we have shown that local scale transformations exist for any θ, act as dynamical symmetries of certain non-local free field theories and appear to be realized as space-time symmetries in some strongly anisotropic critical systems of physical interest.

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